

# Topological string amplitudes for the local $\frac{1}{2}\mathbf{K3}$ surface

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## Abstract

We study topological string amplitudes for the local  $\frac{1}{2}\mathbf{K3}$  surface. We develop a method of computing higher genus amplitudes along the lines of the direct integration formalism, making full use of the Seiberg–Witten curve expressed in terms of modular forms and  $E_8$ -invariant Jacobi forms. The Seiberg–Witten curve was constructed previously for the low-energy effective theory of the non-critical E-string theory in  $\mathbb{R}^4 \times T^2$ . We clarify how the amplitudes are written as polynomials in a finite number of generators expressed in terms of the Seiberg–Witten curve. We determine the coefficients of the polynomials by solving the holomorphic anomaly equation and the gap condition and construct the amplitudes explicitly up to genus three. The results encompass topological string amplitudes for all local del Pezzo surfaces.

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## 1. Introduction

Topological string theory on the local  $\frac{1}{2}$ K3 surface provides us with a unified description of the low-energy effective theory of four-dimensional  $\mathcal{N} = 2$  SU(2) gauge theories [1, 2] and their extensions to five and six dimensions. The local  $\frac{1}{2}$ K3 surface is a non-compact Calabi–Yau threefold in which the  $\frac{1}{2}$ K3 surface appears as a divisor. By blowing down exceptional curves, one can reduce  $\frac{1}{2}$ K3 to any del Pezzo surfaces  $\mathcal{B}_n$  ( $n \leq 8$ ), including  $\mathbb{P}^2$  and  $\mathbb{P}^1 \times \mathbb{P}^1$ . Topological string theory on the local  $\frac{1}{2}$ K3 describes the low-energy effective theory of the six-dimensional (1,0) supersymmetric non-critical E-string theory in  $\mathbb{R}^4 \times T^2$  [3–8]. Similarly, topological string theory on the local  $\mathcal{B}_n$  corresponds to the non-critical  $E_n$  string theory in  $\mathbb{R}^4 \times T^2$  with one of the cycles of the  $T^2$  shrinking to zero size [9, 10]. This theory shares the same moduli space with the five-dimensional  $\mathcal{N} = 1$  SU(2) gauge theory on  $\mathbb{R}^4 \times S^1$  with  $n - 1$  fundamental matters [11–13]. For the toric case ( $n \leq 5$ ), the topological string amplitudes have been well studied. In particular, the all-genus topological string partition function in this case is given by the Nekrasov partition function for the above five-dimensional gauge theory [14–17].

For toric Calabi–Yau threefolds, the construction of topological string amplitudes has been well understood. One can use the topological vertex formalism [18] to construct the all-genus partition function as a sum over partitions on the A-model side. The remodeling B-model conjecture [19], extended from the topological recursion for matrix models [20], enables us to generate the amplitudes recursively with respect to the genus on the B-model side [21, 22]. Indeed, for toric local del Pezzo surfaces, topological string amplitudes have been studied both in the former approach [23] and in the latter approach [24, 25]. For non-toric Calabi–Yau threefolds, however, such a universal prescription is lacking at present. The purpose of this paper is to formulate a method of constructing the topological string amplitudes for the most general local  $\frac{1}{2}$ K3 surface.

A generalization of the topological vertex formalism was proposed [26] and applied to the construction of the topological string partition functions for non-toric local del Pezzo surfaces [27]. (See also [28] for another construction for the local  $\mathcal{B}_6$ .) Remarkably, this formalism enables us to construct the all-genus partition function as a sum over partitions. The partition function in this form is, however, not suitable for obtaining the topological string amplitude at each genus in a closed form. Also these constructions do not seem to apply directly to the case of the general local  $\frac{1}{2}$ K3 surface. On the other hand, one can construct the topological string amplitude at

each genus by solving the holomorphic anomaly equation [29]. Higher genus amplitudes have been constructed explicitly for some special cases with one or two moduli parameters [30–32]. Moreover, a simple, specific form of the holomorphic anomaly equation was proposed for the topological string amplitudes for the local  $\frac{1}{2}\text{K3}$  surface [8, 31]. By solving this equation one can construct higher genus amplitudes for the most general case with manifest affine  $E_8$  symmetry [8, 33]. In this construction, however, the amplitudes are obtained not in a closed form, but rather in the form of an instanton expansion with respect to one of the Kähler moduli parameters.

Recently, Grimm, Klemm, Marino and Weiss proposed the direct integration method [34], which provides us with an efficient way of solving holomorphic anomaly equations. Using this method, one can obtain the amplitudes at each genus in a closed form. The key point of the method is to make use of the fact that the topological string amplitudes can be expressed as polynomials in a finite number of generators [35]. This is achieved by taking account of the symmetry, in particular modular properties of the amplitudes [36]. The method is applicable, in principle, to topological strings on any Calabi–Yau manifold. It has also been applied to the gravitational corrections to Seiberg–Witten theories [34, 37, 38].

There are many examples of non-compact Calabi–Yau threefolds for which the mirror geometries are essentially described by Seiberg–Witten curves. In this case, the symmetry of the topological string amplitudes can naturally be understood in terms of the Seiberg–Witten curve. The Seiberg–Witten curve turns out to be useful to construct the topological string amplitude not only at genus zero, but also at higher genus. All these arguments apply to the local  $\frac{1}{2}\text{K3}$  surface: The mirror geometry in this case is described by the Seiberg–Witten curve for the E-string theory [7, 39]. In particular, the most general form expressed in terms of modular forms and  $E_8$ -invariant Jacobi forms was constructed [39]. Making full use of this Seiberg–Witten curve, we are able to formulate a method of constructing the topological string amplitudes at higher genus in a closed form for the most general local  $\frac{1}{2}\text{K3}$ .

Let us briefly summarize our construction in the following. We first clarify the polynomial structure of the higher genus amplitudes and identify the generators of the polynomials. The generators are expressed in terms of one of the periods and the complex structure modulus of the torus associated with the Seiberg–Witten curve. We elucidate the modular anomaly of the generators, which can be interpreted as the holomorphic anomaly. This enables us to evaluate the holomorphic anomaly of the ansätze for the higher genus amplitudes. Each time we solve the holomorphic

anomaly equation, there appears a holomorphic ambiguity that cannot be fixed by the equation. We fix them by imposing a gap condition. The gap condition for the topological strings on the local  $\frac{1}{2}\text{K3}$  surface has been known [8]. This comes from the geometric property of the local  $\frac{1}{2}\text{K3}$ . Using this method, we construct the amplitudes explicitly up to genus three.

While the basic idea of our construction is the same as that of the direct integration method, our method is rather different from the standard one in appearance. We start from the holomorphic anomaly equation of Hosono–Saito–Takahashi [31] specific to the present model, rather than that of Bershadsky–Cecotti–Ooguri–Vafa (BCOV) [29]. We use our original generators when constructing ansätze for the amplitudes. In terms of these generators the amplitudes can be concisely expressed. Despite these differences, we think both methods are essentially equivalent. Later we show that the amplitudes and the holomorphic anomaly equation can be written in a form akin to what has been obtained for other models by the standard direct integration method [34, 36–38].

As we mentioned in the beginning, the topological string theory on the local  $\frac{1}{2}\text{K3}$  surface encompasses that on all local del Pezzo surfaces. Remarkably, when the topological string amplitudes for the local  $\frac{1}{2}\text{K3}$  are expressed in terms of the Seiberg–Witten curve, their forms are universal to all local del Pezzo surfaces. To obtain the amplitudes for any local del Pezzo surface, we have only to reduce the Seiberg–Witten curve correspondingly [10, 40]. By way of illustration, we present explicit forms of amplitudes for three basic examples, the massless local  $\mathcal{B}_8$ , the local  $\mathbb{P}^2$  and the local  $\mathbb{P}^1 \times \mathbb{P}^1$ .

This paper is organized as follows. In section 2, we review some basic properties of the topological string amplitudes for the local  $\frac{1}{2}\text{K3}$  surface. In section 3, we describe the method of constructing topological string amplitudes for the local  $\frac{1}{2}\text{K3}$  in a closed form. First we review how the topological string amplitude at genus zero is constructed from the Seiberg–Witten curve. We then study the modular anomaly of fundamental quantities and interpret them as the holomorphic anomaly. With these data, we solve the holomorphic anomaly equation at low genus. We make a conjecture on the general structure of the amplitudes, which greatly simplifies the problem of solving the holomorphic anomaly equation. We present two other expressions for the amplitudes and the holomorphic anomaly equation. In particular, the last expression is similar to what is found in the standard direct integration method. In section 4, we study how to reduce our general results to the topological string amplitudes for

all local del Pezzo surfaces. We present explicit forms of amplitudes for three basic examples, the massless local  $\mathcal{B}_8$ , the local  $\mathbb{P}^2$  and the local  $\mathbb{P}^1 \times \mathbb{P}^1$ . Section 5 is devoted to the conclusion and the discussion. In Appendix A, we present explicitly the generators of  $E_8$ -invariant Jacobi forms and the Seiberg–Witten curve for the present model. Appendix B is a collection of derivative formulas. In Appendix C, we present the explicit form of the amplitude at genus three. In Appendix D, we summarize our conventions of special functions.

## 2. Properties of topological string amplitudes for local $\frac{1}{2}\text{K3}$

In this section we review some basic properties of the topological string amplitudes for the local  $\frac{1}{2}\text{K3}$  surface. The reader is referred to references [8, 32] for further details.

The  $\frac{1}{2}\text{K3}$  surface, also known as the rational elliptic surface or the almost del Pezzo surface  $\mathcal{B}_9$ , is obtained by blowing up nine base points of a pencil of cubic curves in  $\mathbb{P}^2$ . The  $\frac{1}{2}\text{K3}$  surface admits an elliptic fibration over  $\mathbb{P}^1$ . A generic  $\frac{1}{2}\text{K3}$  surface has 12 singular fibers, while a generic elliptic K3 surface has 24 singular fibers.

The second homology group  $H_2(\frac{1}{2}\text{K3}, \mathbb{Z})$  is generated by the class of a line in  $\mathbb{P}^2$  and the nine classes of the exceptional curves. With an inner product given by the intersection number,  $H_2(\frac{1}{2}\text{K3}, \mathbb{Z})$  acquires the structure of the ten-dimensional odd unimodular Lorentzian lattice  $\Gamma^{9,1}$  (also denoted by  $\text{I}_{9,1}$ ). The automorphism group of  $\Gamma^{9,1}$  contains the Weyl group of the affine  $E_8$  root system. This property is crucial to our construction of the topological string amplitudes for the local  $\frac{1}{2}\text{K3}$ . It is also useful to note that the lattice decomposes as  $\Gamma^{9,1} = \Gamma^{1,1} \oplus \Gamma_8$ , where  $\Gamma^{1,1}$  is the two-dimensional odd unimodular Lorentzian lattice and  $\Gamma_8$  is the  $E_8$  root lattice.  $\Gamma^{1,1}$  is generated by  $[B], [E]$  with  $[B] \cdot [E] = 1$ ,  $[B] \cdot [B] = -1$ ,  $[E] \cdot [E] = 0$ , where  $[B]$  and  $[E]$  can be viewed as the classes of the base and the fiber of the elliptic fibration. The automorphism group of  $\Gamma_8$  is given by the Weyl group of the  $E_8$  root system, which will be denoted by  $W(E_8)$ .

By a local  $\frac{1}{2}\text{K3}$  surface we mean the total space of the canonical bundle of a generic  $\frac{1}{2}\text{K3}$  surface. It is a non-compact Calabi–Yau threefold. We consider the A-model topological string theory on it. In this paper we let  $F_g$  denote *the instanton part* of the topological string amplitude at genus  $g$ . What we mean by *the instanton part* will be explained soon. We consider the amplitudes in real polarization, namely,  $F_g$  are holomorphic functions. As we will see below, the holomorphic anomaly of the

amplitudes can be read from the modular anomaly.

Let  $F$  denote the all-genus topological string partition function defined as

$$F = \sum_{g=0}^{\infty} F_g x^{2g-2}. \quad (2.1)$$

$F$  can be viewed as the generating function of the Gopakumar–Vafa invariants [41]. By taking account of the  $W(E_8)$  symmetry,  $F$  can be expressed as

$$F(\varphi, \tau, \boldsymbol{\mu}; x) = \sum_{r=0}^{\infty} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \sum_{\boldsymbol{\lambda} \in P_+} \sum_{\boldsymbol{w} \in \mathcal{O}_{\boldsymbol{\lambda}}} N_{n,k,\boldsymbol{\lambda}}^r \sum_{m=1}^{\infty} \frac{1}{m} \left( 2 \sin \frac{mx}{2} \right)^{2r-2} e^{2\pi i m(n\varphi + k\tau + \boldsymbol{w} \cdot \boldsymbol{\mu})}. \quad (2.2)$$

Here  $P_+$  denotes the set of all dominant weights of  $E_8$  and the sum with respect to weights  $\boldsymbol{w}$  is taken over the Weyl orbit of  $\boldsymbol{\lambda}$ .  $\varphi$  and  $\tau$  denote the Kähler moduli corresponding to the base and the fiber of the elliptic fibration, respectively, while  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_8)$  denote the orthogonal coordinates for the complexified root space of  $E_8$ . The Gopakumar–Vafa invariants  $N_{n,k,\boldsymbol{\lambda}}^r$  are integers. They count the BPS multiplicities of the five-dimensional  $\mathcal{N} = 1$  supersymmetric theory obtained by compactifying the M-theory on the local  $\frac{1}{2}$ K3 surface. This five-dimensional theory is identified with the effective theory of the six-dimensional E-string theory on  $\mathbb{R}^5 \times S^1$ .

We defined  $F_g$  as *the instanton part*, which means that  $F_g$  is expanded as

$$F_g(\varphi, \tau, \boldsymbol{\mu}) = \sum_{n=1}^{\infty} Z_{g,n}(\tau, \boldsymbol{\mu}) e^{2\pi i n \varphi} \quad (2.3)$$

and does not contain any polynomial (including constant) term in  $\varphi$ . From the point of view of the E-string theory,  $Z_n := e^{-\pi i n \tau} Z_{0,n}$  is the BPS partition function of the  $n$ -wound E-strings [5, 8].  $Z_n$  is also interpreted as the partition function of  $\mathcal{N} = 4$  U( $n$ ) topological Yang–Mills theory on  $\frac{1}{2}$ K3 [8]. Throughout this paper, we refer to this  $F_g$  as the topological string amplitude at genus  $g$ .

In general, it is rather hard to solve a topological string model with ten Kähler moduli parameters, in particular when the target space is not a local toric Calabi–Yau threefold. In the present case, however, one can make full use of the symmetry to construct the amplitudes. It turns out that  $F$  is fully characterized by the symmetry, the holomorphic anomaly equation and the gap condition.

Let us start with the symmetry. Due to the automorphism of the homology lattice of  $\frac{1}{2}$ K3, the partition function exhibits the affine  $E_8$  symmetry. Moreover,

it possesses good modular properties in  $\tau$ . It is known that  $Z_{g,n}$  has the following structure [32]

$$Z_{g,n}(\tau, \boldsymbol{\mu}) = \frac{T_{g,n}(\tau, \boldsymbol{\mu})}{[\prod_{k=1}^{\infty} (1 - q^k)]^{12n}}, \quad (2.4)$$

where

$$q = e^{2\pi i \tau}. \quad (2.5)$$

$T_{g,n}$  is an  $W(E_8)$ -invariant quasi-Jacobi form of weight  $2g - 2 + 6n$  and index  $n$ . The reader is referred to Appendix A for the basic properties of the  $W(E_8)$ -invariant Jacobi form. By  $W(E_8)$ -invariant *quasi*-Jacobi forms we mean those which are generated by the generators of the ordinary  $W(E_8)$ -invariant Jacobi forms and the Eisenstein series  $E_2(\tau)$ .

$E_2(\tau)$  is not strictly a modular form, as it transforms as

$$E_2\left(-\frac{1}{\tau}\right) = \tau^2 \left(E_2(\tau) + \frac{6}{\pi i \tau}\right). \quad (2.6)$$

However, the following non-holomorphic function

$$\hat{E}_2(\tau, \bar{\tau}) := E_2(\tau) + \frac{6}{\pi i (\tau - \bar{\tau})} \quad (2.7)$$

transforms as a modular form of weight 2

$$\hat{E}_2\left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}\right) = \tau^2 \hat{E}_2(\tau, \bar{\tau}). \quad (2.8)$$

By replacing all  $E_2(\tau)$  by  $\hat{E}_2(\tau, \bar{\tau})$ , the amplitudes  $F_g$  transforms as a modular function of weight  $2g - 2$  at the cost of loosing holomorphicity. This non-holomorphicity is regarded as the holomorphic anomaly of the amplitudes. In other words, the modular/holomorphic anomaly of the amplitudes always appears through  $E_2$ . For later convenience, we introduce a normalized notation  $\xi := \frac{1}{24}E_2$  and let

$$\partial_\xi = 24\partial_{E_2} \quad (2.9)$$

measure the holomorphic anomaly. We also introduce a normalized variable  $\phi = 2\pi i \varphi + \phi_0$ , so that

$$\partial_\phi = \frac{1}{2\pi i} \partial_\varphi. \quad (2.10)$$

The precise relation between  $\phi$  and  $\varphi$  will be given in section 3. Throughout this paper we hold  $\tau$  and  $\boldsymbol{\mu}$  constant when we take partial derivatives with respect to  $\xi$

and  $\phi$ . In terms of these normalized variables, the holomorphic anomaly equation for the partition function  $F$  is written as [31, 32]

$$\partial_\xi e^F = x^2 \partial_\phi (\partial_\phi + 1) e^F. \quad (2.11)$$

By expanding the equation in  $x$ , it becomes a set of recursive equations

$$\partial_\xi F_g = \partial_\phi^2 F_{g-1} + \partial_\phi F_{g-1} + \sum_{h=0}^g \partial_\phi F_h \partial_\phi F_{g-h}. \quad (2.12)$$

The equation for  $g = 0$  should be understood with  $F_{-1} = 0$ . In terms of  $Z_{g,n}$ , the holomorphic anomaly equations read

$$\partial_\xi Z_{g,n} = n(n+1)Z_{g-1,n} + \sum_{h=0}^g \sum_{k=1}^{n-1} k(n-k)Z_{h,k}Z_{g-h,n-k}. \quad (2.13)$$

Again, the equation for  $g = 0$  should be understood with  $Z_{-1,n} = 0$ .

The above form of holomorphic anomaly equation was first proposed for  $g = 0$  [8] and later extended for general  $g$  [31]. While in [31] the equation was applied to the case with  $\mu = 0$ , it is also valid for the cases with nonzero  $\mu$  [32, 33]. The relation to the BCOV holomorphic anomaly equations [29] has also been discussed [33].

As the holomorphic anomaly equation is a differential equation, one needs to fix the integration constant, i.e. the holomorphic ambiguity, at each genus. For the present model, it is known that the following gap condition can be used for this purpose

$$F = \sum_{n=1}^{\infty} e^{2\pi i n \varphi} \left( \frac{1}{n(2 \sin \frac{n\varphi}{2})^2} + \mathcal{O}(q^n) \right). \quad (2.14)$$

This condition is equivalent to the following constraint on the Gopakumar–Vafa invariants

$$N_{n,k,\lambda}^g = 0 \quad \text{for } k < n \quad \text{except } N_{1,0,0}^0 = 1. \quad (2.15)$$

This follows from the geometric structure of the local  $\frac{1}{2}\text{K3}$  [8]. In terms of  $Z_{g,n}$  the gap condition reads

$$Z_{g,n} = \beta_g n^{2g-3} + \mathcal{O}(q^n), \quad (2.16)$$

where  $\beta_g$  are rational numbers defined by the following expansion

$$\begin{aligned} \sum_{g=0}^{\infty} \beta_g x^{2g} &= \frac{x^2}{4 \sin^2 \frac{x}{2}} \\ &= 1 + \frac{1}{12}x^2 + \frac{1}{240}x^4 + \frac{1}{6048}x^6 + \mathcal{O}(x^8). \end{aligned} \quad (2.17)$$



It has been checked [8,32,33], at least for low  $g$  and  $n$ , that  $Z_{g,n}$  can be determined uniquely by the symmetry (2.4), the holomorphic anomaly equations (2.13) and the gap conditions (2.16). Assuming that this holds for general  $g$  and  $n$ , we will develop a method of constructing  $F_g$  in a closed form in the next section.

### 3. Closed expressions for amplitudes

#### 3.1. Genus zero amplitude and instanton expansion

It is known that the genus zero amplitude  $F_0$  for the local  $\frac{1}{2}$ K3 surface is obtained as the prepotential associated with the Seiberg–Witten curve of the following form

$$y^2 = 4x^3 - fx - g, \quad (3.1)$$

with

$$f = \sum_{j=0}^4 a_j u^{4-j}, \quad g = \sum_{j=0}^6 b_j u^{6-j}. \quad (3.2)$$

Actually, a Seiberg–Witten curve of this form itself describes an elliptic fibration of the  $\frac{1}{2}$ K3 surface. It can be viewed as a sort of local mirror symmetry between one  $\frac{1}{2}$ K3 and another  $\frac{1}{2}$ K3 [8, 32]. We present the explicit form of the Seiberg–Witten curve in Appendix A. It was determined in [39] so that the instanton expansion of the prepotential correctly reproduces  $Z_{0,n}$  at low  $n$  calculated by the method of [8], which we summarized in the last section.

Let us recall how the prepotential is obtained from the Seiberg–Witten curve of the above general form. Given the Seiberg–Witten curve (3.1), the vev of the scalar component of the  $\mathcal{N} = 2$  vector multiplet is expressed as

$$\phi = -\frac{1}{2\pi} \int du \oint_{\alpha} \frac{dx}{y}, \quad (3.3)$$

where  $\alpha$  is one of the fundamental cycles of the curve. The complexified gauge coupling constant  $\tilde{\tau}$  is given by the complex structure modulus of the Seiberg–Witten curve. On the other hand,  $\tilde{\tau}$  is given by the second derivative of the prepotential. In terms of the instanton part  $F_0$  of the prepotential,  $\tilde{\tau}$  is expressed as

$$\tilde{\tau} = \tau + \frac{i}{2\pi} \partial_{\phi}^2 F_0, \quad (3.4)$$

where  $\tau$  is the bare gauge coupling constant. By solving these relations, one obtains the prepotential from the Seiberg–Witten curve.

The practical calculation can be organized as follows [10, 39, 42]. Since the present Seiberg–Witten curve is elliptic, one can make full use of the explicit map between

an elliptic curve and a torus. Let  $(2\pi\omega, 2\pi\omega\tilde{\tau})$  denote the fundamental periods of the torus. The map from the torus to the elliptic curve in the Weierstrass form (3.1) is given in terms of the Weierstrass  $\wp$ -function by

$$x = \wp(z; 2\pi\omega, 2\pi\omega\tilde{\tau}), \quad y = \partial_z \wp(z; 2\pi\omega, 2\pi\omega\tilde{\tau}). \quad (3.5)$$

The coefficients of the elliptic curve and the periods of the torus are related as

$$f = \frac{1}{12} \frac{\tilde{E}_4}{\omega^4}, \quad g = \frac{1}{216} \frac{\tilde{E}_6}{\omega^6}, \quad (3.6)$$

where we use the notation

$$\tilde{E}_{2n} := E_{2n}(\tilde{\tau}). \quad (3.7)$$

One can express  $\tilde{\tau}$  and  $\omega$  in terms of the Seiberg–Witten curve by inverting the modular functions. First, we eliminate  $\omega$  from the two equations (3.6) by taking the ratio of  $f^3/g^2$ . Equivalently, we can look at the  $j$ -invariant and expand it in  $u^{-1}$  as

$$\begin{aligned} \frac{1}{\tilde{j}} &= \frac{\tilde{E}_4^3 - \tilde{E}_6^2}{1728\tilde{E}_4^3} = \frac{f^3 - 27g^2}{1728f^3} \\ &= \frac{1}{j} - \frac{E_6 b_1}{4E_4^3} \frac{1}{u} + \mathcal{O}\left(\frac{1}{u^2}\right). \end{aligned} \quad (3.8)$$

Here we have used  $a_0 = \frac{1}{12}E_4$ ,  $b_0 = \frac{1}{216}E_6$ ,  $a_1 = 0$ .<sup>1</sup> On the other hand, the  $j$ -invariant has the following expansion

$$\tilde{j} = \frac{1}{\tilde{q}} + 744 + 196884\tilde{q} + \mathcal{O}(\tilde{q}^2), \quad \tilde{q} = e^{2\pi i\tilde{\tau}}. \quad (3.9)$$

Inverting this expansion and using (3.8), we obtain the expansion of  $\tilde{\tau}$  in  $u^{-1}$ . By introducing the notation

$$t := 2\pi i(\tilde{\tau} - \tau), \quad (3.10)$$

the expansion is expressed as

$$t = -\frac{E_4 b_1}{4\Delta} \frac{1}{u} + \left( \frac{E_6 a_2}{48\Delta} - \frac{E_4 b_2}{4\Delta} - \frac{E_4(E_4 E_2 + 5E_6)b_1^2}{192\Delta^2} \right) \frac{1}{u^2} + \mathcal{O}\left(\frac{1}{u^3}\right). \quad (3.11)$$

Substituting this into (3.6), one obtains the expansion of  $\omega$  in  $u^{-1}$ . We choose the sign of  $\omega$  in such a way that  $\omega$  is expanded as

$$\omega = \frac{1}{u} - \frac{(E_4 E_2 - E_6)b_1}{48\Delta} \frac{1}{u^2} + \mathcal{O}\left(\frac{1}{u^3}\right). \quad (3.12)$$

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<sup>1</sup> This is the convention of the Seiberg–Witten curve adopted in [39]. By suitable rescaling and shift of variables one can always recast a generic Seiberg–Witten curve of the form (3.1), (3.2) as this form without loss of generality.

Integrating this by  $u$ , one obtains  $\phi$ . We define  $\phi$  with the normalization

$$\phi := - \int \omega du \quad (3.13)$$

so that  $e^\phi$  has the following expansion

$$e^\phi = \frac{1}{u} - \frac{(E_4 E_2 - E_6) b_1}{48 \Delta} \frac{1}{u^2} + \mathcal{O}\left(\frac{1}{u^3}\right). \quad (3.14)$$

Inverting this relation, we have

$$\frac{1}{u} = e^\phi + \frac{(E_4 E_2 - E_6) b_1}{48 \Delta} e^{2\phi} + \mathcal{O}(e^{3\phi}). \quad (3.15)$$

Substituting this into (3.11), we obtain

$$t = -\frac{E_4 b_1}{4 \Delta} e^\phi + \left( \frac{E_6 a_2}{48 \Delta} - \frac{E_4 b_2}{4 \Delta} - \frac{E_4 (E_4 E_2 + 2 E_6) b_1^2}{96 \Delta^2} \right) e^{2\phi} + \mathcal{O}(e^{3\phi}). \quad (3.16)$$

Similarly, from (3.12) and (3.15) we obtain

$$\begin{aligned} \ln \omega = \phi + & \left( \frac{(E_6 E_2 - E_4^2) a_2}{1152 \Delta} - \frac{(E_4 E_2 - E_6) b_2}{96 \Delta} \right. \\ & \left. + \frac{(-2 E_4^2 E_2^2 - 8 E_6 E_4 E_2 + 5 E_4^3 + 5 E_6^2) b_1^2}{9216 \Delta^2} \right) e^{2\phi} + \mathcal{O}(e^{3\phi}), \end{aligned} \quad (3.17)$$

which will be used later. As explained in the beginning, the instanton part of the prepotential is given by

$$F_0 = -\partial_\phi^{-2} t. \quad (3.18)$$

Here  $\partial_\phi^{-1}$  denotes integration with respect to  $\phi$  of a power series in  $e^\phi$ .

This prepotential is identified with the genus zero amplitude  $F_0$  for the local  $\frac{1}{2}$ K3 surface. The bare gauge coupling  $\tau$  is interpreted as the Kähler modulus  $\tau$  of the original  $\frac{1}{2}$ K3. The scalar vev  $\phi$  is identified with the Kähler modulus  $\varphi$  with [39]

$$e^\phi = -q \left[ \prod_{k=1}^{\infty} (1 - q^k) \right]^{12} e^{2\pi i \varphi}. \quad (3.19)$$

Taking this into account,  $F_0$  is expanded as

$$\begin{aligned} F_0 = & -\frac{E_4 b_1}{4 \left[ \prod_{k=1}^{\infty} (1 - q^k) \right]^{12}} e^{2\pi i \varphi} \\ & + \frac{-2 E_6 \Delta a_2 + 24 E_4 \Delta b_2 + (E_4^2 E_2 + 2 E_6 E_4) b_1^2}{384 \left[ \prod_{k=1}^{\infty} (1 - q^k) \right]^{24}} e^{4\pi i \varphi} + \mathcal{O}(e^{6\pi i \varphi}). \end{aligned} \quad (3.20)$$

By substituting the coefficients  $a_n, b_n$  of the Seiberg–Witten curve presented in Appendix A, one obtains the genus zero amplitude for the local  $\frac{1}{2}$ K3 as a series expansion in  $e^{2\pi i \varphi}$  up to any desired order.

### 3.2. Modular anomaly

The Seiberg–Witten curve transforms as a  $W(E_8)$ -invariant Jacobi form (see Appendix A). On the other hand, the genus zero amplitude  $F_0$  contains  $E_2$  and therefore exhibits the modular anomaly. The  $E_2$ 's appear when one expands the  $j$ -invariant  $j(\tilde{\tau})$  around  $\tilde{\tau} = \tau$ . Thus, the modulus  $\tilde{\tau}$  and the period  $\omega$  of the Seiberg–Witten curve do exhibit the modular anomaly when expanded in  $u^{-1}$ . In [42], the modular anomaly of  $\tilde{\tau}$  and  $\omega$  was studied in the course of proving the holomorphic anomaly equation for the genus zero amplitude. Extending the analysis, here we study the modular anomaly of various quantities derived from the Seiberg–Witten curve. We will use this to solve the holomorphic anomaly equation for higher genus amplitudes.

As mentioned above, the Seiberg–Witten curve transforms as a Jacobi form. This means that the modulus  $\tilde{\tau}(u, \tau, \boldsymbol{\mu})$  of the curve transforms in precisely the same way as  $\tau$  does under the action of  $\text{SL}(2, \mathbb{Z})$ . It then follows that  $t = 2\pi i(\tilde{\tau} - \tau)$  is invariant under  $\tau \rightarrow \tau + 1$ ,  $\tilde{\tau} \rightarrow \tilde{\tau} + 1$ , while it transforms as

$$\frac{1}{t} \rightarrow \tau^2 \left( \frac{1}{t} + \frac{1}{2\pi i \tau} \right) \quad \text{for} \quad \tau \rightarrow -\frac{1}{\tau}, \quad \tilde{\tau} \rightarrow -\frac{1}{\tilde{\tau}}. \quad (3.21)$$

This anomalous behavior is expected since there appear  $E_2$ 's in the coefficients of the expansion (3.11). Moreover, one finds that the transformation of  $t^{-1}$  is very similar to that of  $E_2$  as in (2.6). This suggests that  $t^{-1}$  depends on  $E_2$  as

$$t^{-1} = \frac{1}{12} E_2 + (\text{modular function of weight } 2). \quad (3.22)$$

One can explicitly check this using the series expansion (3.11). Let us express it as

$$(\partial_\xi t^{-1})_u = 2, \quad (3.23)$$

or

$$(\partial_\xi t)_u = -2t^2. \quad (3.24)$$

Here  $(\partial_\xi t)_u$  denotes the partial derivative of  $t$  with respect to  $\xi$ , holding  $u$  constant.

Next let us consider modular properties of the combination

$$\omega t = \left( \frac{\tilde{E}_4}{12f} \right)^{1/4} t. \quad (3.25)$$

We have used (3.6). One can see that this transforms as a modular form of weight 4 in  $\tau$ , since the constituents transform as

$$\tilde{E}_4 \rightarrow \tilde{\tau}^4 \tilde{E}_4, \quad f \rightarrow \tau^{-20} f, \quad t \rightarrow \tilde{\tau}^{-1} \tau^{-1} t \quad (3.26)$$

under the S-transformation  $\tau \rightarrow -1/\tau$ ,  $\tilde{\tau} \rightarrow -1/\tilde{\tau}$ . This means that the combination  $\omega t$  is free of modular anomaly, namely

$$(\partial_\xi(\omega t))_u = 0. \quad (3.27)$$

Using (3.24), one obtains

$$(\partial_\xi \omega)_u = 2\omega t. \quad (3.28)$$

Furthermore, combining (3.28) with (3.13) one obtains

$$(\partial_\xi \phi)_u = 2\partial_\phi^{-1} t, \quad (3.29)$$

$$(\partial_\xi^2 \phi)_u = 0. \quad (3.30)$$

Based on these formulas and (3.6), one can evaluate modular anomaly of various quantities. We present a list of formulas in Appendix B.

So far in this subsection, we regard  $u$  and  $\xi$  as independent variables and we take the derivative  $\partial_\xi$  holding  $u$  constant. Let us say we are in the  $(u, \xi)$  frame. On the other hand, the holomorphic anomaly equations (2.12) are given in the  $(\phi, \xi)$  frame. It is useful to see how expressions in these frames are transformed into each other. The derivative of a function  $A$  with respect to  $\xi$  is transformed between these frames by the simple chain rule

$$(\partial_\xi A)_\phi = -(\partial_\xi \phi)_u (\partial_\phi A)_\xi + (\partial_\xi A)_u, \quad (3.31)$$

$$= -2(\partial_\phi^{-1} t) (\partial_\phi A)_\xi + (\partial_\xi A)_u. \quad (3.32)$$

We have used (3.29) in the second equality. We sometimes omit the subscript  $\xi$ , as we always hold  $\xi$  constant when we take derivatives  $\partial_u$  and  $\partial_\phi$ . Applying this formula to  $\partial_\phi F_0 = -\partial_\phi^{-1} t$  and using (3.29), (3.30), we see that

$$(\partial_\xi (\partial_\phi F_0))_\phi = 2 (\partial_\phi F_0) (\partial_\phi^2 F_0). \quad (3.33)$$

By integrating both sides by  $\phi$ , we obtain the holomorphic anomaly equation (2.12) at  $g = 0$

$$\partial_\xi F_0 = (\partial_\phi F_0)^2. \quad (3.34)$$

### 3.3. Higher genus amplitudes

The expression (2.12) of the holomorphic anomaly equation is not convenient for practical purposes, since derivatives of  $F_g$  appear on both sides of the equation.

Using  $\partial_\phi F_0 = -\partial_\phi^{-1}t$  and the chain rule (3.32), one can rewrite the equation into the following recursive form

$$(\partial_\xi F_g)_u = \partial_\phi^2 F_{g-1} + \partial_\phi F_{g-1} + \sum_{h=1}^{g-1} \partial_\phi F_h \partial_\phi F_{g-h} \quad (3.35)$$

for  $g \geq 1$ . In the following, we solve this equation and construct  $F_g$  for low  $g$ .

Let us first consider the case of  $g = 1$ . In this case, the equation simply reads

$$\begin{aligned} (\partial_\xi F_1)_u &= \partial_\phi^2 F_0 + \partial_\phi F_0 \\ &= -t - \partial_\phi^{-1}t. \end{aligned} \quad (3.36)$$

With the help of the derivative formulas (B.5)–(B.9), one immediately finds a solution of the following form

$$F_1 = c_1 \ln \omega - \left( \frac{c_1}{12} + \frac{1}{24} \right) \ln \tilde{\Delta} - \frac{1}{2} \phi + f_1(\tau). \quad (3.37)$$

The constant  $c_1$  and the function  $f_1(\tau)$  can be determined by the condition that  $F_1$  takes the form (2.3), namely it does not contain any polynomial term in  $\phi$ . From (3.16) and (3.17) we see that

$$\ln \tilde{\Delta} = \ln \Delta + \mathcal{O}(e^{2\pi i \varphi}), \quad \ln \omega = \phi + \mathcal{O}(e^{4\pi i \varphi}). \quad (3.38)$$

Using these we can determine the unknowns as  $c_1 = 1/2$ ,  $f_1 = (\ln \Delta)/12$  and obtain

$$F_1 = \frac{1}{2} \ln \omega - \frac{1}{12} \ln \tilde{\Delta} + \frac{1}{12} \ln \Delta - \frac{1}{2} \phi. \quad (3.39)$$

While this is not a rigorous derivation, we checked that the above form is the correct answer. Combined with (3.15)–(3.17) and (3.19), the above expression correctly reproduces  $Z_{1,n}$ , which we explicitly calculated up to  $n = 5$  using the method explained in the last section. It also reproduces the result for  $\boldsymbol{\mu} = \mathbf{0}$  presented in [33]. Note that a similar expression has been known for four-dimensional Seiberg–Witten theories [17, 38].

To compute amplitudes for  $g \geq 2$  by solving (3.35), we point out an interesting fact that the term  $-\phi/2$  in  $F_1$  precisely cancels the linear term  $\partial_\phi F_{g-1}$  on the r.h.s. of (3.35). Therefore, if we introduce the notation

$$\mathcal{F}_1 = F_1 + \frac{1}{2} \phi, \quad \mathcal{F}_2 = F_2 + \frac{1}{96} E_2, \quad \mathcal{F}_g = F_g \quad \text{for } g \geq 3, \quad (3.40)$$

the holomorphic anomaly equation (3.35) turns into a very simple form

$$(\partial_\xi \mathcal{F}_g)_u = \partial_\phi^2 \mathcal{F}_{g-1} + \sum_{h=1}^{g-1} \partial_\phi \mathcal{F}_h \partial_\phi \mathcal{F}_{g-h} \quad (3.41)$$

for  $g \geq 2$ . Note that this form has been already presented in [38] in the case of four-dimensional SU(2) Seiberg–Witten theories. It is natural that the holomorphic anomaly equation takes the same form in the present case, since the definition (3.13) of  $\phi$  through the Seiberg–Witten curve is common in both cases.

Based on this simple form, let us search for the amplitude at  $g = 2$ . The equation (3.41) in this case reads

$$\begin{aligned} (\partial_\xi \mathcal{F}_2)_u &= \partial_\phi^2 \mathcal{F}_1 + (\partial_\phi \mathcal{F}_1)^2 \\ &= \frac{1}{2} \partial_\phi^2 \ln \omega + \frac{1}{4} (\partial_\phi \ln \omega)^2 - \frac{1}{12} \tilde{E}_2 \partial_\phi t \partial_\phi \ln \omega - \frac{1}{12} \tilde{E}_2 \partial_\phi^2 t + \frac{1}{144} \tilde{E}_4 (\partial_\phi t)^2. \end{aligned} \quad (3.42)$$

Note that the rightmost expression is a polynomial of (quasi-)modular forms  $\tilde{E}_{2k}$  and derivatives  $\partial_\phi^m \ln \omega$ ,  $\partial_\phi^n t$ . The polynomial is constrained so that each term contains two  $\partial_\phi$ 's and is of weight 0. Note that after every  $E_2$  is replaced by  $\hat{E}_2$ , a modular function in  $\tilde{\tau}$  transforms as that in  $\tau$  with the same weight. Thus, the weights of the generators of the polynomial read

$$[\tilde{E}_{2k}] = 2k, \quad [\partial_\phi^m \ln \omega] = 0, \quad [\partial_\phi^n t] = -2. \quad (3.43)$$

We see from (3.42) that  $\mathcal{F}_2$  is of weight 2, since  $\xi$  is of weight 2. Let us make an ansatz that  $\mathcal{F}_2$  has the same polynomial structure as (3.42), namely a polynomial of  $\tilde{E}_{2k}$ ,  $\partial_\phi^m \ln \omega$ ,  $\partial_\phi^n t$  with two  $\partial_\phi$ 's. Explicitly, the ansatz reads

$$\begin{aligned} \mathcal{F}_2 &= c_1 \tilde{E}_2 \partial_\phi^2 \ln \omega + c_2 \tilde{E}_2 (\partial_\phi \ln \omega)^2 + (c_3 \tilde{E}_2^2 + c_4 \tilde{E}_4) \partial_\phi t \partial_\phi \ln \omega \\ &\quad + (c_5 \tilde{E}_2^2 + c_6 \tilde{E}_4) \partial_\phi^2 t + (c_7 \tilde{E}_2^3 + c_8 \tilde{E}_4 \tilde{E}_2 + c_9 \tilde{E}_6) (\partial_\phi t)^2. \end{aligned} \quad (3.44)$$

Substituting this ansatz into (3.42), one can partly determine the coefficients  $c_j$ . The derivatives of the generators with respect to  $\xi$  are summarized in Appendix B. One has to be careful when taking derivatives of  $\partial_\phi^n \ln \omega$  and  $\partial_\phi^n t$  with respect to  $\xi$ . We differentiate them in the  $(u, \xi)$  frame, where  $\partial_\xi$  and  $\partial_\phi$  do not commute. The explicit forms of these derivatives for general  $n$  are given in (B.10), (B.11), which can be shown by using the chain rule (3.32).

The holomorphic anomaly equation (3.42) reduces the number of undetermined parameters to three. These remaining parameters can be fixed by the condition that

$F_2$  takes the form (2.3) and by the gap conditions (2.16) at  $n = 1, 2$ . In the end, one obtains

$$\begin{aligned} \mathcal{F}_2 = & \frac{1}{48} \tilde{E}_2 \partial_\phi^2 \ln \omega + \frac{1}{96} \tilde{E}_2 (\partial_\phi \ln \omega)^2 - \frac{1}{576} (\tilde{E}_2^2 - \tilde{E}_4) \partial_\phi t \partial_\phi \ln \omega \\ & - \frac{1}{1920} (5\tilde{E}_2^2 + 3\tilde{E}_4) \partial_\phi^2 t - \frac{1}{207360} (35\tilde{E}_2^3 + 51\tilde{E}_4 \tilde{E}_2 - 86\tilde{E}_6) (\partial_\phi t)^2. \end{aligned} \quad (3.45)$$

In the same way, we are able to determine the amplitude at genus three: The most general ansatz for  $\mathcal{F}_3$  is written with 68 unknown parameters. The holomorphic anomaly equation gives 45 relations and leaves 23 undetermined parameters. These are fixed completely by the condition that  $F_3$  takes the form (2.3) and by the gap conditions (2.16) up to  $n = 4$ . The explicit form of  $F_3$  is presented in Appendix C. We checked for low  $n$  that  $Z_{g,n}$  calculated from the above obtained  $\mathcal{F}_2, \mathcal{F}_3$  coincide with the results obtained by the method described in section 2. In the next section we will also reproduce the higher genus amplitudes for local del Pezzo surfaces from these results, which serves as another consistency check.

Based on the above explicit construction of  $\mathcal{F}_g$  at low genera, we propose the following conjecture:

$\mathcal{F}_g \ (g \geq 2) \text{ is a polynomial in } \tilde{E}_{2k}, \partial_\phi^m \ln \omega, \partial_\phi^n t \ (k = 1, 2, 3, m, n \in \mathbb{Z}_{>0}),$   
in which each term contains  $2g - 2$   $\partial_\phi$ 's and is of weight  $2g - 2$ .

(3.46)

Note that the form of the polynomial is no longer unique for  $g \geq 4$ , since not all of  $\partial_\phi^m \ln \omega, \partial_\phi^n t$  are independent. Actually, they are finitely generated. This can be seen as follows. Recall that  $f, g$  are polynomials of degree 4, 6 in  $u$ , respectively. Since  $\partial_u = -\omega \partial_\phi$ , this means that

$$(\omega \partial_\phi)^k \frac{\tilde{E}_4}{\omega^4} = 0, \quad \text{for } k > 4, \quad (3.47)$$

$$(\omega \partial_\phi)^k \frac{\tilde{E}_6}{\omega^6} = 0, \quad \text{for } k > 6. \quad (3.48)$$

These relations give rise to non-trivial relations among the derivatives  $\partial_\phi^m \ln \omega, \partial_\phi^n t$ . Using these relations, one can express all  $\partial_\phi^m \ln \omega$  and  $\partial_\phi^n t$  with  $m, n \in \mathbb{Z}_{>0}$  in terms of those with  $m = 1, \dots, 6$  and  $n = 1, \dots, 4$  and  $\tilde{E}_2, \tilde{E}_4, \tilde{E}_6$ . Therefore, by assuming the above conjecture,  $\mathcal{F}_g$  can also be expressed in terms of these generators. In this expression  $\mathcal{F}_g \ (g \geq 4)$  is still a polynomial in  $\partial_\phi^m \ln \omega$  and  $\partial_\phi^n t$ , but becomes a rational function in  $\tilde{E}_{2k}$ . In subsection 3.5 we will introduce another expression in which  $\mathcal{F}_g$  is written indeed as a polynomial of a finite number of generators.



The above conjecture provides us with a systematic construction of the ansatz for general  $\mathcal{F}_g$ . We expect that the holomorphic anomaly equations and the gap conditions will be sufficient for completely fixing the form of  $\mathcal{F}_g$ . We have seen that this is indeed the case for  $g = 2, 3$ . We have not checked it for  $g \geq 4$ .

### 3.4. Expression in $(u, \xi)$ frame

The equation (3.41) looks somewhat irregular, as the l.h.s. is written in the  $(u, \xi)$  frame while the r.h.s. is in the  $(\phi, \xi)$  frame. For practical purposes, this is actually a convenient form since the derivatives of  $\mathcal{F}_g$  are assembled in a single term in the  $(u, \xi)$  frame while the gap condition can be explicitly expressed in the  $(\phi, \xi)$  frame. On the other hand, it is also useful to express the equation entirely in the  $(u, \xi)$  frame. By using  $\partial_\phi = -\frac{1}{\omega}\partial_u$ , (3.41) can be rewritten as

$$(\partial_\xi \mathcal{F}_g)_u = \frac{1}{\omega^2} \left( \partial_u^2 \mathcal{F}_{g-1} - \partial_u \ln \omega \partial_u \mathcal{F}_{g-1} + \sum_{h=1}^{g-1} \partial_u \mathcal{F}_h \partial_u \mathcal{F}_{g-h} \right) \quad (3.49)$$

for  $g \geq 2$ . In this frame, it is easier to take derivatives with respect to  $\xi$ , while the gap condition cannot be expressed in a simple manner.  $\mathcal{F}_2$  is expressed as

$$\begin{aligned} \mathcal{F}_2 = \frac{1}{\omega^2} & \left( \frac{1}{48} \tilde{E}_2 \partial_u^2 \ln \omega - \frac{1}{96} \tilde{E}_2 (\partial_u \ln \omega)^2 + \frac{1}{5760} (5\tilde{E}_2^2 + 19\tilde{E}_4) \partial_u t \partial_u \ln \omega \right. \\ & \left. - \frac{1}{1920} (5\tilde{E}_2^2 + 3\tilde{E}_4) \partial_u^2 t - \frac{1}{207360} (35\tilde{E}_2^3 + 51\tilde{E}_4 \tilde{E}_2 - 86\tilde{E}_6) (\partial_u t)^2 \right), \end{aligned} \quad (3.50)$$

which is almost as simple as the previous expression (3.45).

### 3.5. Expression in direct integration style

There is another interesting expression of the topological string amplitudes and the holomorphic anomaly equation: One can express the amplitudes directly in terms of the coefficients of the Seiberg–Witten curve. To see this, let us start with studying the transformation rules for the generators.

From (3.6) and

$$D := f^3 - 27g^2 = \frac{\tilde{\Delta}}{\omega^{12}}, \quad (3.51)$$

we see that

$$\begin{aligned}\partial_u \ln f &= \partial_u \ln \tilde{E}_4 - 4\partial_u \ln \omega \\ &= \frac{1}{3} \left( \tilde{E}_2 - \frac{\tilde{E}_6}{\tilde{E}_4} \right) \partial_u t - 4\partial_u \ln \omega,\end{aligned}\tag{3.52}$$

$$\begin{aligned}\partial_u \ln D &= \partial_u \ln \tilde{\Delta} - 12\partial_u \ln \omega \\ &= \tilde{E}_2 \partial_u t - 12\partial_u \ln \omega.\end{aligned}\tag{3.53}$$

Solving these relations, one obtains

$$\partial_u t = \frac{1}{\omega^2} \frac{-6fg' + 9f'g}{2D},\tag{3.54}$$

$$\partial_u \ln \omega = \frac{(-2fg' + 3f'g)X + (-2f^2f' + 36gg')}{8D},\tag{3.55}$$

where

$$X := \frac{\tilde{E}_2}{\omega^2}.\tag{3.56}$$

The derivative of  $X$  in  $u$  is computed as

$$\partial_u X = \frac{(2fg' - 3f'g)X^2 + (4f^2f' - 72gg')X + (24f^2g' - 36ff'g)}{8D}.\tag{3.57}$$

With the help of these relations and  $\partial_\phi = -\frac{1}{\omega}\partial_u$ , it is straightforward to express higher derivatives of  $t$  and  $\ln \omega$  in terms of  $X$  and  $f^{(m)}(u)$ ,  $g^{(n)}(u)$ . Note that  $\tilde{E}_4, \tilde{E}_6$  can also be rewritten in terms of  $f, g, \omega$  by using (3.6). After all,  $\mathcal{F}_2$  is expressed as

$$\begin{aligned}\mathcal{F}_2 &= \frac{1}{92160D^2} \left( (100f^2g'^2 - 300ff'gg' + 225f'^2g^2)X^3 \right. \\ &\quad + (240f^4g'' - 780f^3f'g' - 360f^3f''g + 990f^2f'^2g \\ &\quad - 6480fg^2g'' + 11880fgg'^2 - 14580f'g^2g' + 9720f''g^3)X^2 \\ &\quad + (-480f^5f'' + 420f^4f'^2 + 8640f^3gg'' + 10080f^3g'^2 - 54000f^2f'gg' \\ &\quad + 12960f^2f''g^2 + 29160ff'^2g^2 - 233280g^3g'' + 213840g^2g'^2)X \\ &\quad + (5184f^5g'' - 12816f^4f'g' - 7776f^4f''g + 15336f^3f'^2g - 139968f^2g^2g'' \\ &\quad \left. + 258336f^2gg'^2 - 428976ff'g^2g' + 209952ff''g^3 + 167184f'^2g^3) \right). \end{aligned}\tag{3.58}$$

Note that  $\omega$  does not appear explicitly in this expression. The same type of expression for  $F_3$  is immediately obtained by rewriting the result in Appendix C. We do not present its lengthy expression here, but the calculation is straightforward.

The holomorphic anomaly equations can be written in a form more suited to the above expression. Observe that when  $\mathcal{F}_g$  are expressed as in (3.58), holomorphic

anomalies appear only through  $X$ . Hence, one can simply replace  $\partial_\xi$  by

$$\partial_\xi = (\partial_\xi X)_u \partial_X = \frac{24}{\omega^2} \partial_X \quad (3.59)$$

in the holomorphic anomaly equation (3.49). For  $g = 2$ , the equation is now written as

$$24\partial_X \mathcal{F}_2 = \partial_u^2 \mathcal{F}_1 - \partial_u \ln \omega \partial_u \mathcal{F}_1 + (\partial_u \mathcal{F}_1)^2. \quad (3.60)$$

By using

$$\begin{aligned} \partial_u \mathcal{F}_1 &= -\frac{1}{2} \partial_u \ln \omega - \frac{1}{12} \partial_u \ln D \\ &= \frac{(2fg' - 3f'g)X + (-2f^2f' + 36gg')}{16D} \end{aligned} \quad (3.61)$$

and (3.54), (3.55), (3.57), one can evaluate the r.h.s. of (3.60) as a quadratic polynomial in  $X$ . Then integrating directly both hands of (3.60) in  $X$ , one obtains (3.58) up to the ‘constant’ part in  $X$ . For  $g \geq 3$ , let us introduce the notation

$$\check{\mathcal{F}}_1 = \mathcal{F}_1 - \frac{1}{2} \ln \omega, \quad \check{\mathcal{F}}_g = \mathcal{F}_g \quad \text{for } g \geq 2. \quad (3.62)$$

The holomorphic anomaly equations can then be written again in a very simple form

$$24\partial_X \check{\mathcal{F}}_g = \partial_u^2 \check{\mathcal{F}}_{g-1} + \sum_{h=1}^{g-1} \partial_u \check{\mathcal{F}}_h \partial_u \check{\mathcal{F}}_{g-h}, \quad g \geq 3. \quad (3.63)$$

Regarding the explicit form of  $\mathcal{F}_g$  at  $g = 2, 3$  and the above equation, we present our conjecture on the structure of the amplitudes in another form:  $\mathcal{F}_g$  ( $g \geq 2$ ) can be expressed as

$$\mathcal{F}_g = \frac{1}{D^{2g-2}} \sum_{k=0}^{3g-3} P_{g,k}[\partial_u^{2g-2}, f, g] X^k. \quad (3.64)$$

Here  $P_{g,k}[\partial_u^{2g-2}, f, g]$  denotes a polynomial of  $\partial_u^m f, \partial_u^n g$  ( $m, n \in \mathbb{Z}_{\geq 0}$ ) in which each term contains  $2g-2$   $\partial_u$ ’s. The form of the polynomial is constrained so that  $\mathcal{F}_g$  transforms as a  $W(E_8)$ -invariant quasi-Jacobi form of index 0. Note that the constituents of the amplitudes are of the following indices (see Appendix A):

$$[X] = 2, \quad [f] = 4, \quad [g] = 6, \quad [D] = 12, \quad [\partial_u] = -1. \quad (3.65)$$

Recall that  $f, g$  are polynomials of degree 4, 6 in  $u$ , respectively. Therefore, in this expression it is manifest that  $\mathcal{F}_g$  is a polynomial of a finite number of generators, namely,

$$\frac{1}{D}, \quad X, \quad \partial_u^m f, \quad \partial_u^n g, \quad m = 0, \dots, 4, \quad n = 0, \dots, 6. \quad (3.66)$$

We find that the above structure of the amplitudes and the holomorphic anomaly equation is akin to what are obtained for other models by the direct integration method [34,37,38]. While we have taken a different path from the standard approach, both constructions should be essentially equivalent.

#### 4. Topological string amplitudes for local del Pezzo surfaces

The topological string amplitudes for the local  $\frac{1}{2}$ K3 surface encompass those for all local del Pezzo surfaces. In this section we see how the former reduce to the latter. In fact, when the topological string amplitudes for the local  $\frac{1}{2}$ K3 are expressed in terms of the Seiberg–Witten curve, the forms of them are universal to all local del Pezzo surfaces. We obtain the amplitudes for any local del Pezzo surface by merely replacing the Seiberg–Witten curve with the corresponding one. The mirror pair of the local del Pezzo surface  $\mathcal{B}_n$  is given by the Seiberg–Witten curve for the five-dimensional  $E_n$  strings [10]. It is also easy to reduce the most general Seiberg–Witten curve to that for any del Pezzo surface [10,40,43]. We first discuss the general cases and then present explicit forms of amplitudes for three basic examples, the massless local  $\mathcal{B}_8$ , the local  $\mathbb{P}^2$  and the local  $\mathbb{P}^1 \times \mathbb{P}^1$ .

##### 4.1. General cases

The Seiberg–Witten curve for the local  $\mathcal{B}_8$  is obtained from that for the local  $\frac{1}{2}$ K3 by simply taking the limit  $q \rightarrow 0$ . Curves for the other local  $\mathcal{B}_n$  ( $n \leq 7$ ) are immediately obtained by a suitable rescaling [10,40]. The construction of the topological string amplitudes from the Seiberg–Witten curve is essentially the same as in the case of the local  $\frac{1}{2}$ K3. In particular, the mirror map between  $u$  and  $\phi$  for  $\mathcal{B}_n$  ( $n \leq 8$ ) is simply given by the  $q \rightarrow 0$  limit of (3.15).

Below we present the minor modifications needed for the local  $\mathcal{B}_n$  ( $n \leq 8$ ). The instanton parts of the topological string amplitudes at  $g = 0, 1$  are slightly modified as follows

$$F_0 = -\partial_\phi^{-2} t + \frac{9-n}{6} \phi^3, \quad (4.1)$$

$$F_1 = \frac{1}{2} \ln \omega - \frac{1}{12} \ln \tilde{\Delta} - \frac{1}{2} \phi + \frac{9-n}{12} \phi \quad (4.2)$$

with

$$t := 2\pi i \tilde{\tau} \quad (4.3)$$

instead of (3.10). Expressions for higher genus amplitudes  $\mathcal{F}_g$  ( $g \geq 2$ ) hold as they stand, where  $\mathcal{F}_g$  are now related to  $F_g$  as

$$\mathcal{F}_1 = F_1 + \frac{1}{2}\phi, \quad \mathcal{F}_2 = F_2 + \frac{1}{96}, \quad \mathcal{F}_g = F_g \quad \text{for } g \geq 3. \quad (4.4)$$

We also need to modify the relation (3.19) between  $\phi$  and  $\varphi$ , since it is no longer valid in the limit  $q = 0$ . Instead of (3.19), we identify them by

$$e^\phi = -e^{2\pi i \varphi}. \quad (4.5)$$

#### 4.2. Massless local $\mathcal{B}_8$

As an illustration we first consider the case of local  $\mathcal{B}_8$  with  $\mu = \mathbf{0}$ . In this case the corresponding Seiberg–Witten curve is extremely simple. The coefficients are given by

$$f = \frac{1}{12}u^4, \quad g = \frac{1}{216}u^6 - 4u^5. \quad (4.6)$$

The amplitudes at  $g = 0, 1$  are given by (4.1), (4.2) with  $n = 8$ . By substituting the above  $f, g$  into (3.58) one obtains

$$\begin{aligned} \mathcal{F}_2 = \frac{1}{207360u^4(u-432)^2} & \left( 25X^3 + 15u(-25u + 6048)X^2 \right. \\ & \left. + 75u^2(29u^2 - 22464u + 5225472)X + u^5(335u - 273888) \right). \end{aligned} \quad (4.7)$$

Similarly, from the expression of  $F_3$  in Appendix C, one obtains

$$\begin{aligned} \mathcal{F}_3 = \frac{1}{5016453120u^8(u-432)^4} & \left( 525X^6 - 8400u^2X^5 \right. \\ & + 315u^2(175u^2 + 5184u + 5225472)X^4 \\ & + 560u^3(-325u^3 + 18360u^2 - 89859456u + 11851370496)X^3 \\ & + 63u^4(4625u^4 - 5008896u^3 + 8491143168u^2 \\ & - 2300402073600u + 260052929740800)X^2 \\ & + 672u^7(-325u^3 + 284796u^2 - 623837376u + 7054387200)X \\ & \left. + u^8(61775u^4 - 96755904u^3 + 219325750272u^2 \right. \\ & \left. + 15910182715392u + 9788763779629056) \right). \end{aligned} \quad (4.8)$$

From these expressions one can compute Gopakumar–Vafa invariants. The instanton expansions in this case read

$$\frac{1}{u} = e^\phi - 60e^{2\phi} - 1530e^{3\phi} - 274160e^{4\phi} - 50519055e^{5\phi} + \mathcal{O}(e^{6\phi}), \quad (4.9)$$

$$\omega = e^\phi + 5130e^{3\phi} + 1347520e^{4\phi} + 372046365e^{5\phi} + \mathcal{O}(e^{6\phi}), \quad (4.10)$$

$$t = \phi + 252e^\phi + 36882e^{2\phi} + 7637736e^{3\phi} + 1828258569e^{4\phi} + \mathcal{O}(e^{5\phi}). \quad (4.11)$$

$N_n^r$	$n$	1	2	3	4	5	$\dots$
$r$							
0		252	-9252	848628	-114265008	18958064400	
1		-2	760	-246790	76413833	-23436186176	
2		0	-4	30464	-26631112	16150498760	
3		0	0	-1548	5889840	-7785768630	
$\vdots$							$\ddots$

Table 1: Gopakumar–Vafa invariants for the massless local  $\mathcal{B}_8$ .

The Gopakumar–Vafa invariants are computed by recasting  $F$  as

$$\sum_{g=0}^{\infty} F_g x^{2g-2} = \sum_{r=0}^{\infty} \sum_{n=1}^{\infty} N_n^r \sum_{m=1}^{\infty} \frac{1}{m} \left( 2 \sin \frac{mx}{2} \right)^{2r-2} e^{2\pi i m n \varphi}. \quad (4.12)$$

We present the Gopakumar–Vafa invariants  $N_n^r$  at low degrees in Table 1. This reproduces the known result, for example found in [5, 32].<sup>2</sup> Moreover, it is easy to compute  $N_n^r$  up to arbitrary large degree of  $n$ , as we now have the exact form of the amplitudes  $\mathcal{F}_g$ .

We have performed the expansion around the large volume point  $u = \infty$  to compute the Gopakumar–Vafa invariants, but we could expand the amplitudes at arbitrary  $u$ . It would be interesting to study the behavior of the amplitudes around the other points such as the orbifold point, as in [24].

### 4.3. Local $\mathbb{P}^2$

The coefficients of the Seiberg–Witten curve are given by

$$f = \frac{1}{12}u^4 - 2u, \quad g = \frac{1}{216}u^6 - \frac{1}{6}u^3 + 1. \quad (4.13)$$

The amplitudes at  $g = 0, 1$  are given by (4.1), (4.2) with  $n = 0$ . By substituting the above  $f, g$  into (3.58) one obtains

$$\mathcal{F}_2 = \frac{75X^3 - 165u^2X^2 + 125u^4X + 9(5u^6 - 464u^3 + 6192)}{7680(u^3 - 27)^2}. \quad (4.14)$$

---

<sup>2</sup> The Gopakumar–Vafa invariants  $N_n^r$  at  $r = 1$  and the instanton numbers  $\tilde{N}_n^g$  for  $g = 1$  curves found in [5, 32] are related by  $N_n^1 = \sum_{k|n} \tilde{N}_{(n/k)}^1$  [32].

$N_n^r$	$n$	1	2	3	4	5	...
$r$							
0		3	-6	27	-192	1695	
1		0	0	-10	231	-4452	
2		0	0	0	-102	5430	
3		0	0	0	15	-3672	
$\vdots$							$\ddots$

Table 2: Gopakumar–Vafa invariants for the local  $\mathbb{P}^2$ .

Similarly, from the expression of  $F_3$  in Appendix C, one obtains

$$\begin{aligned} \mathcal{F}_3 = & \frac{1}{20643840(u^3 - 27)^4} \Big( 14175X^6 - 75600u^2X^5 + 315u(533u^3 + 3024)X^4 \\ & - 560(355u^6 + 6750u^3 + 8748)X^3 \\ & + 21u^2(6305u^6 + 257472u^3 + 1181952)X^2 \\ & - 672u(70u^9 + 5007u^6 + 49086u^3 + 34992)X \\ & + 6965u^{12} + 774992u^9 + 13201920u^6 + 27993600u^3 + 20155392 \Big). \end{aligned} \quad (4.15)$$

The instanton expansions in this case are given by

$$\frac{1}{u} = e^\phi - 2e^{4\phi} - e^{7\phi} - 20e^{10\phi} - 177e^{13\phi} + \mathcal{O}(e^{16\phi}), \quad (4.16)$$

$$\omega = e^\phi + 4e^{4\phi} + 41e^{7\phi} + 520e^{10\phi} + 7275e^{13\phi} + \mathcal{O}(e^{16\phi}), \quad (4.17)$$

$$t = 9\phi + 27e^{3\phi} + \frac{405}{2}e^{6\phi} + 2196e^{9\phi} + \frac{110997}{4}e^{12\phi} + \mathcal{O}(e^{15\phi}). \quad (4.18)$$

The all-genus topological string partition function can be expressed as

$$\sum_{g=0}^{\infty} F_g x^{2g-2} = \sum_{r=0}^{\infty} \sum_{n=1}^{\infty} N_n^r \sum_{m=1}^{\infty} \frac{1}{m} \left( 2 \sin \frac{mx}{2} \right)^{2r-2} Q^{mn}, \quad (4.19)$$

where

$$Q = e^{6\pi i \varphi} = -e^{3\phi}. \quad (4.20)$$

Table 2 shows the Gopakumar–Vafa invariants  $N_n^r$  at low  $r$  and  $n$ . This coincides with the known result (see [30, 44], for example) of the Gopakumar–Vafa invariants for local  $\mathbb{P}^2$ .

#### 4.4. Local $\mathbb{P}^1 \times \mathbb{P}^1$

The coefficients of the Seiberg–Witten curve are given by

$$\begin{aligned} f &= \frac{1}{12}u^4 - \frac{2}{3}\chi u^2 + \frac{4}{3}\chi^2 - 4, \\ g &= \frac{1}{216}u^6 - \frac{1}{18}\chi u^4 + \left(\frac{2}{9}\chi^2 - \frac{1}{3}\right)u^2 + \left(-\frac{8}{27}\chi^3 + \frac{4}{3}\chi\right), \end{aligned} \quad (4.21)$$

where

$$\chi = e^{2\pi i\mu} + e^{-2\pi i\mu}. \quad (4.22)$$

The amplitudes at  $g = 0, 1$  are given by (4.1), (4.2) with  $n = 1$ . By substituting the above  $f, g$  into (3.58) one obtains

$$\begin{aligned} \mathcal{F}_2 &= \frac{1}{12960(u^2 - 4\chi + 8)^2(u^2 - 4\chi - 8)^2} \Big( 100u^2 X^3 \\ &\quad + 120(-2u^4 + 5\chi u^2 + 12\chi^2 - 48) X^2 \\ &\quad + 15(13u^6 - 80\chi u^4 + (16\chi^2 + 768)u^2 + 384\chi^3 - 1536\chi) X \\ &\quad + 8(10u^8 - 201u^6\chi + (1452\chi^2 - 2808)u^4 \\ &\quad + (-4528\chi^3 + 20304\chi)u^2 + 5184\chi^4 - 36288\chi^2 + 62208) \Big). \end{aligned} \quad (4.23)$$

We do not present the explicit form of  $\mathcal{F}_3$  since it is slightly lengthy, but the calculation is straightforward. Instanton expansions in this case read

$$\frac{1}{u} = e^\phi - \chi e^{3\phi} + (\chi^2 - 3)e^{5\phi} + (-\chi^3 + \chi)e^{7\phi} + \mathcal{O}(e^{9\phi}), \quad (4.24)$$

$$\omega = e^\phi + \chi e^{3\phi} + (\chi^2 + 9)e^{5\phi} + (\chi^3 + 43\chi)e^{7\phi} + \mathcal{O}(e^{9\phi}), \quad (4.25)$$

$$t = 8\phi + 8\chi e^{2\phi} + (4\chi^2 + 56)e^{4\phi} + \left(\frac{8}{3}\chi^3 + 208\chi\right)e^{6\phi} + \mathcal{O}(e^{8\phi}). \quad (4.26)$$

The all-genus topological string partition function can be expressed as

$$\sum_{g=0}^{\infty} F_g x^{2g-2} = \sum_{r=0}^{\infty} \sum_{n_1, n_2=0}^{\infty} N_{n_1, n_2}^r \sum_{m=1}^{\infty} \frac{1}{m} \left(2 \sin \frac{mx}{2}\right)^{2r-2} Q_1^{mn_1} Q_2^{mn_2}, \quad (4.27)$$

where

$$Q_1 = e^{2\pi i(2\varphi + \mu)}, \quad Q_2 = e^{2\pi i(2\varphi - \mu)}. \quad (4.28)$$

We checked that  $N_{n_1, n_2}^r$  coincide with the known data of the Gopakumar–Vafa invariants for the local  $\mathbb{P}^1 \times \mathbb{P}^1$  (see [44], for example).



## 5. Conclusion and discussion

In this paper we have developed a general method of computing topological string amplitudes for the local  $\frac{1}{2}\text{K3}$  surface. We have demonstrated that the amplitudes can be concisely expressed in terms of the Seiberg–Witten curve which manifestly exhibits good modular properties and the affine  $E_8$  Weyl group invariance. We have clarified the general structure of the amplitudes: The amplitudes at  $g = 0, 1$  are given in (3.18), (3.39), while higher genus amplitudes  $\mathcal{F}_g$  ( $g \geq 2$ ) are written as a polynomial of generators expressed in terms the Seiberg–Witten curve. In particular, in the  $(u, \xi)$  frame, the amplitudes are polynomials in a finite number of generators. Given the structure, one can determine the coefficients of the polynomials by solving the holomorphic anomaly equation and the gap condition. We have explicitly computed the form of the amplitudes for  $g = 2, 3$ . We have also found that the holomorphic anomaly equation takes a very simple form if we adopt notations in which the amplitudes at low genus are slightly modified.

The topological strings on the local  $\frac{1}{2}\text{K3}$  surface encompass those on all local del Pezzo surfaces. We have elucidated how to reduce the amplitudes to those for the local del Pezzo surfaces. By way of illustration, we have explicitly constructed the amplitudes for three simple cases. These amplitudes correctly reproduce the known Gopakumar–Vafa invariants.

There are several directions for further investigation. We believe that the conjecture (3.46), (3.64) on the structure of the amplitudes hold for general  $g$ . It is of importance to check this further and prove them. Another expectation to be clarified is that with the above conjecture the holomorphic anomaly equations and the gap conditions may be sufficient to determine the amplitudes at arbitrary high genus. It is known that the gap condition is indeed sufficient at least for local del Pezzo surfaces with one or two moduli parameters [45]. We expect that the same sort of argument will apply to the case of the most general local  $\frac{1}{2}\text{K3}$  surface.

We have started from the holomorphic anomaly equation (2.12) of [31] which is specific to the present model rather than the general one of BCOV [29]. For practical purposes, the former equation is easier to deal with. Nevertheless, it would still be of interest to clarify the relation between these two equations and see how our construction fits in the general scheme of the topological string theory.

The direct integration method has been applied to the four-dimensional  $\text{SU}(2)$  Seiberg–Witten theories with matters [37, 38]. We know from Nekrasov partition functions that by taking a certain limit topological string amplitudes on the toric

del Pezzo surfaces reproduce the prepotential and the gravitational corrections of the four-dimensional theories. It is interesting to see how our general formulas reproduce those results. The cases of non-toric local del Pezzo surfaces are of particular interest. In terms of the Seiberg–Witten curves, we know how the four dimensional  $SU(2)$  theories with an  $E_n$  global symmetry [46–48] are reproduced from the five dimensional ones [10,39]. It would be interesting to construct the gravitational corrections to these four-dimensional theories with an  $E_n$  flavor symmetry.

The topological recursion [20], or more specifically the remodeling B-model conjecture [19], is a powerful method of computing topological string amplitudes. This method is free of the holomorphic ambiguity and also computes the open string amplitudes. On the other hand, for the moment this method has not yet been applied to topological strings on non-toric Calabi–Yau manifolds. It would be very interesting if our expressions for the amplitudes  $\mathcal{F}_g$  can be derived by a method similar to the topological recursion.

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## A. Seiberg–Witten curve for E-string theory

The low energy effective theory of the E-string theory in  $\mathbb{R}^4 \times T^2$  is described as SU(2) Seiberg–Witten theory with nine parameters,  $\tau$  and  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_8)$ .  $\tau$  is regarded as the bare gauge coupling and  $\boldsymbol{\mu}$  are the masses of fundamental matters. The theory possesses an  $E_8$  flavor symmetry and the Weyl group  $W(E_8)$  acts on  $\boldsymbol{\mu}$  as an automorphism. On the other hand, from the point of view of the six-dimensional theory,  $\tau$  is the modulus of the  $T^2$  in the 5,6-directions and  $\boldsymbol{\mu}$  are interpreted as Wilson lines along these directions. The theory therefore admits modular properties in  $\tau$  and double periodicity in  $\boldsymbol{\mu}$ . These symmetries become manifest if we express the dependence on these parameters through  $W(E_8)$ -invariant Jacobi forms.

### A.1. $W(E_8)$ -invariant Jacobi forms

Let  $\varphi_{k,m}(\tau, \boldsymbol{\mu})$  denote  $W(E_8)$ -invariant Jacobi forms of weight  $k$  and index  $m$ . They are holomorphic in  $\tau$  ( $\text{Im } \tau > 0$ ),  $\boldsymbol{\mu} \in \mathbb{C}^8$  and satisfy the following properties [49, 50]:

i) Weyl invariance

$$\varphi_{k,m}(\tau, w(\boldsymbol{\mu})) = \varphi_{k,m}(\tau, \boldsymbol{\mu}), \quad w \in W(E_8). \quad (\text{A.1})$$

ii) Quasi-periodicity

$$\varphi_{k,m}(\tau, \boldsymbol{\mu} + \mathbf{v} + \tau \mathbf{w}) = e^{-m\pi i(\tau \mathbf{w}^2 + 2\boldsymbol{\mu} \cdot \mathbf{w})} \varphi_{k,m}(\tau, \boldsymbol{\mu}), \quad \mathbf{v}, \mathbf{w} \in \Gamma_8. \quad (\text{A.2})$$

iii) Modular properties

$$\varphi_{k,m}\left(\frac{a\tau + b}{c\tau + d}, \frac{\boldsymbol{\mu}}{c\tau + d}\right) = (c\tau + d)^k \exp\left(m\pi i \frac{c}{c\tau + d} \boldsymbol{\mu}^2\right) \varphi_{k,m}(\tau, \boldsymbol{\mu}). \quad (\text{A.3})$$

iv)  $\varphi_{k,m}(\tau, \boldsymbol{\mu})$  admit a Fourier expansion as

$$\varphi_{k,m}(\tau, \boldsymbol{\mu}) = \sum_{l=0}^{\infty} \sum_{\substack{\mathbf{v} \in \Gamma_8 \\ \mathbf{v}^2 \leq 2ml}} c(l, \mathbf{v}) e^{2\pi i(l\tau + \mathbf{v} \cdot \boldsymbol{\mu})}. \quad (\text{A.4})$$

Here  $\Gamma_8$  is the  $E_8$  root lattice and  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ . Note that in this convention the index  $m$  coincides with the level of the affine  $E_8$  Lie algebra.

Among others, the most fundamental  $W(E_8)$ -invariant Jacobi form is the theta function associated with the lattice  $\Gamma_8$

$$\Theta(\tau, \boldsymbol{\mu}) = \sum_{\mathbf{w} \in \Gamma^8} \exp(\pi i \tau \mathbf{w}^2 + 2\pi i \boldsymbol{\mu} \cdot \mathbf{w}) = \frac{1}{2} \sum_{k=1}^4 \prod_{j=1}^8 \vartheta_k(\mu_j, \tau). \quad (\text{A.5})$$

One can see from the properties of the Jacobi theta functions that  $\Theta(\tau, \boldsymbol{\mu})$  is of weight 4 and index 1. Jacobi forms of higher indices can be constructed from  $\Theta(\tau, \boldsymbol{\mu})$  as follows.

To construct general  $W(E_8)$ -invariant Jacobi forms, we introduce functions

$$\begin{aligned} e_1(\tau) &= \frac{1}{12} (\vartheta_3(\tau)^4 + \vartheta_4(\tau)^4), \\ e_2(\tau) &= \frac{1}{12} (\vartheta_2(\tau)^4 - \vartheta_4(\tau)^4), \\ e_3(\tau) &= \frac{1}{12} (-\vartheta_2(\tau)^4 - \vartheta_3(\tau)^4) \end{aligned} \quad (\text{A.6})$$

and

$$h(\tau) = \vartheta_3(2\tau)\vartheta_3(6\tau) + \vartheta_2(2\tau)\vartheta_2(6\tau). \quad (\text{A.7})$$

Let us then define the following nine  $W(E_8)$ -invariant Jacobi forms

$$\begin{aligned} A_1(\tau, \boldsymbol{\mu}) &= \Theta(\tau, \boldsymbol{\mu}), \quad A_2(\tau, \boldsymbol{\mu}) = \frac{8}{9} \mathcal{H}\{\Theta(2\tau, 2\boldsymbol{\mu})\}, \quad A_3(\tau, \boldsymbol{\mu}) = \frac{27}{28} \mathcal{H}\{\Theta(3\tau, 3\boldsymbol{\mu})\}, \\ A_4(\tau, \boldsymbol{\mu}) &= \Theta(\tau, 2\boldsymbol{\mu}), \quad A_5(\tau, \boldsymbol{\mu}) = \frac{125}{126} \mathcal{H}\{\Theta(5\tau, 5\boldsymbol{\mu})\}, \\ B_2(\tau, \boldsymbol{\mu}) &= \frac{32}{5} \mathcal{H}\{e_1(\tau)\Theta(2\tau, 2\boldsymbol{\mu})\}, \quad B_3(\tau, \boldsymbol{\mu}) = \frac{81}{80} \mathcal{H}\{h(\tau)^2\Theta(3\tau, 3\boldsymbol{\mu})\}, \\ B_4(\tau, \boldsymbol{\mu}) &= \frac{16}{15} \mathcal{H}\{\vartheta_4(2\tau)^4\Theta(4\tau, 4\boldsymbol{\mu})\}, \quad B_6(\tau, \boldsymbol{\mu}) = \frac{9}{10} \mathcal{H}\{h(\tau)^2\Theta(6\tau, 6\boldsymbol{\mu})\}. \end{aligned} \quad (\text{A.8})$$

Here  $\mathcal{H}\{\cdot\}$  denotes the sum of all possible distinct  $\text{SL}(2, \mathbb{Z})$  transforms of the argument. Explicitly, they read

$$\begin{aligned} A_1(\tau, \boldsymbol{\mu}) &= \Theta(\tau, \boldsymbol{\mu}), \quad A_4(\tau, \boldsymbol{\mu}) = \Theta(\tau, 2\boldsymbol{\mu}), \\ A_n(\tau, \boldsymbol{\mu}) &= \frac{n^3}{n^3+1} \left( \Theta(n\tau, n\boldsymbol{\mu}) + \frac{1}{n^4} \sum_{k=0}^{n-1} \Theta\left(\frac{\tau+k}{n}, \boldsymbol{\mu}\right) \right), \quad n = 2, 3, 5, \\ B_2(\tau, \boldsymbol{\mu}) &= \frac{32}{5} \left( e_1(\tau)\Theta(2\tau, 2\boldsymbol{\mu}) + \frac{1}{2^4} e_3(\tau)\Theta\left(\frac{\tau}{2}, \boldsymbol{\mu}\right) + \frac{1}{2^4} e_2(\tau)\Theta\left(\frac{\tau+1}{2}, \boldsymbol{\mu}\right) \right), \\ B_3(\tau, \boldsymbol{\mu}) &= \frac{81}{80} \left( h(\tau)^2\Theta(3\tau, 3\boldsymbol{\mu}) - \frac{1}{3^5} \sum_{k=0}^2 h\left(\frac{\tau+k}{3}\right)^2 \Theta\left(\frac{\tau+k}{3}, \boldsymbol{\mu}\right) \right), \\ B_4(\tau, \boldsymbol{\mu}) &= \frac{16}{15} \left( \vartheta_4(2\tau)^4\Theta(4\tau, 4\boldsymbol{\mu}) - \frac{1}{2^4} \vartheta_4(2\tau)^4\Theta\left(\tau + \frac{1}{2}, 2\boldsymbol{\mu}\right) \right. \\ &\quad \left. - \frac{1}{2^{2 \cdot 4^4}} \sum_{k=0}^3 \vartheta_2\left(\frac{\tau+k}{2}\right)^4 \Theta\left(\frac{\tau+k}{4}, \boldsymbol{\mu}\right) \right), \\ B_6(\tau, \boldsymbol{\mu}) &= \frac{9}{10} \left( h(\tau)^2\Theta(6\tau, 6\boldsymbol{\mu}) + \frac{1}{2^4} \sum_{k=0}^1 h(\tau+k)^2\Theta\left(\frac{3\tau+3k}{2}, 3\boldsymbol{\mu}\right) \right. \\ &\quad \left. - \frac{1}{3 \cdot 3^4} \sum_{k=0}^2 h\left(\frac{\tau+k}{3}\right)^2 \Theta\left(\frac{2\tau+2k}{3}, 2\boldsymbol{\mu}\right) \right. \\ &\quad \left. - \frac{1}{3 \cdot 6^4} \sum_{k=0}^5 h\left(\frac{\tau+k}{3}\right)^2 \Theta\left(\frac{\tau+k}{6}, \boldsymbol{\mu}\right) \right). \end{aligned} \quad (\text{A.9})$$

$A_n, B_n$  are of index  $n$  and weight 4, 6, respectively. If we set  $\boldsymbol{\mu} = \mathbf{0}$ , these Jacobi forms reduce to ordinary modular forms. We have determined the normalization of  $A_n, B_n$  so that they reduce to the Eisenstein series

$$A_n(\tau, \mathbf{0}) = E_4(\tau), \quad B_n(\tau, \mathbf{0}) = E_6(\tau). \quad (\text{A.10})$$

The above nine  $A_n, B_n$  are chosen in such a way that characters of all fundamental representations of the affine  $E_8$  algebra are expressed as polynomials of  $A_n, B_n$  with some coefficient functions in  $\tau$ , and vice versa.<sup>3</sup> Consequently, any  $W(E_8)$ -invariant Jacobi form that is made of characters of integrable representations of the affine  $E_8$  algebra is expressed in terms of the above  $A_n, B_n$ . Together with the generators of modular forms, which we formally express as

$$A_0(\tau, \boldsymbol{\mu}) = E_4(\tau), \quad B_0(\tau, \boldsymbol{\mu}) = E_6(\tau), \quad (\text{A.11})$$

$A_n, B_n$  generate a ring of  $W(E_8)$ -invariant Jacobi forms that are made of characters of integrable representations of the affine  $E_8$  algebra.

## A.2. Seiberg–Witten curve

The Seiberg–Witten curve for the  $E$ -string theory was constructed in [39]. Here we present the same curve expressed in terms of the  $W(E_8)$ -invariant Jacobi forms introduced above:

$$y^2 = 4x^3 - fx - g, \quad (\text{A.12})$$

$$f = \sum_{j=0}^4 a_j u^{4-j}, \quad g = \sum_{j=0}^6 b_j u^{6-j}, \quad (\text{A.13})$$

$$\begin{aligned} a_0 &= \frac{1}{12} A_0, & a_1 &= 0, & a_2 &= \frac{6}{E_4 \Delta} (-A_0 A_2 + A_1^2), \\ a_3 &= \frac{1}{9 E_4^2 \Delta^2} (-7 A_0^2 B_0 A_3 - 20 A_0^3 B_3 - 9 A_0 B_0 A_1 A_2 + 30 A_0^2 A_1 B_2 + 6 B_0 A_1^3), \\ a_4 &= \frac{1}{864 E_4^3 \Delta^3} \left( (A_0^6 - A_0^3 B_0^2) A_4 + (56 A_0^5 - 56 A_0^2 B_0^2) A_1 A_3 - 27 A_0^5 A_2^2 \right. \\ &\quad \left. - 90 A_0^3 B_0 A_2 B_2 - 75 A_0^4 B_2^2 + (180 A_0^4 - 36 A_0 B_0^2) A_1^2 A_2 \right. \\ &\quad \left. + 240 A_0^2 B_0 A_1^2 B_2 + (-210 A_0^3 + 18 B_0^2) A_1^4 \right), \\ b_0 &= \frac{1}{216} B_0, & b_1 &= -\frac{4}{E_4} A_1, & b_2 &= \frac{5}{6 E_4^2 \Delta} (A_0^2 B_2 - B_0 A_1^2), \\ b_3 &= \frac{1}{108 E_4^3 \Delta^2} \left( -7 A_0^5 A_3 - 20 A_0^3 B_0 B_3 \right. \\ &\quad \left. - 9 A_0^4 A_1 A_2 + 30 A_0^2 B_0 A_1 B_2 + (16 A_0^3 - 10 B_0^2) A_1^3 \right), \end{aligned}$$

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<sup>3</sup> There are alternative choice of them. For instance, one can take  $\frac{256}{45} \mathcal{H}\{e_1(\tau) \Theta(4\tau, 4\boldsymbol{\mu})\}$  instead of  $B_4$  and/or  $\frac{54}{55} \mathcal{H}\{h(2\tau)^2 \Theta(6\tau, 6\boldsymbol{\mu})\}$  instead of  $B_6$ .

$$\begin{aligned}
b_4 &= \frac{1}{1728E_4^4\Delta^3} \left( (-5A_0^7 + 5A_0^4B_0^2)B_4 + (80A_0^6 - 80A_0^3B_0^2)A_1B_3 \right. \\
&\quad + 9A_0^5B_0A_2^2 + 30A_0^6A_2B_2 + 25A_0^4B_0B_2^2 - 48B_0A_0^4A_1^2A_2 \\
&\quad \left. + (-140A_0^5 + 60A_0^2B_0^2)A_1^2B_2 + (74A_0^3B_0 - 10B_0^3)A_1^4 \right), \\
b_5 &= \frac{1}{72E_4^5\Delta^3} \left( (-21A_0^7 + 21A_0^4B_0^2)A_5 - 294A_0^6A_2A_3 - 770A_0^4B_0B_2A_3 \right. \\
&\quad - 840A_0^4B_0A_2B_3 - 2200A_0^5B_2B_3 + 168A_0^5A_1^2A_3 + 480B_0A_0^3A_1^2B_3 \\
&\quad - 621A_0^5A_1A_2^2 + 3525A_0^4A_1B_2^2 + 1224A_0^4A_1^3A_2 - 240A_0^2B_0A_1^3B_2 \\
&\quad \left. + (-456A_0^3 + 24B_0^2)A_1^5 \right), \\
b_6 &= \frac{1}{13436928E_4^6\Delta^5} \left( (-20A_0^{12} + 40A_0^9B_0^2 - 20A_0^6B_0^4)B_6 \right. \\
&\quad + (-189A_0^{10}B_0 + 378A_0^7B_0^3 - 189A_0^4B_0^5)A_1A_5 \\
&\quad + (-9A_0^{10}B_0 + 9A_0^7B_0^3)A_2A_4 + (-15A_0^{11} + 15A_0^8B_0^2)B_2A_4 \\
&\quad + (-180A_0^{11} + 180A_0^8B_0^2)A_2B_4 + (-300A_0^9B_0 + 300A_0^6B_0^3)B_2B_4 \\
&\quad + (22A_0^9B_0 - 22A_0^6B_0^3)A_1^2A_4 + (150A_0^{10} + 120A_0^7B_0^2 - 270A_0^4B_0^4)A_1^2B_4 \\
&\quad + (196A_0^{10}B_0 - 196A_0^7B_0^3)A_3^2 + (1120A_0^{11} - 1120A_0^8B_0^2)A_3B_3 \\
&\quad + (1600A_0^9B_0 - 1600A_0^6B_0^3)B_3^2 + (-2982A_0^9B_0 + 2982A_0^6B_0^3)A_1A_2A_3 \\
&\quad + (-2520A_0^{10} - 4410A_0^7B_0^2 + 6930A_0^4B_0^4)A_1B_2A_3 \\
&\quad + (3360A_0^{10} - 10920A_0^7B_0^2 + 7560A_0^4B_0^4)A_1A_2B_3 \\
&\quad + (-19800A_0^8B_0 + 19800A_0^5B_0^3)A_1B_2B_3 + (2016A_0^8B_0 - 2016A_0^5B_0^3)A_1^3A_3 \\
&\quad + (-5920A_0^9 + 7360A_0^6B_0^2 - 1440A_0^3B_0^4)A_1^3B_3 + (405A_0^9B_0 + 162A_0^6B_0^3)A_2^3 \\
&\quad + (1215A_0^{10} + 1620A_0^7B_0^2)A_2^2B_2 + 4725A_0^8B_0A_2B_2^2 \\
&\quad + (1125A_0^9 + 1500A_0^6B_0^2)B_2^3 + (-9477A_0^8B_0 + 5103A_0^5B_0^3)A_1^2A_2^2 \\
&\quad + (-9180A_0^9 - 5400A_0^6B_0^2)A_1^2A_2B_2 + (20925A_0^7B_0 - 33075A_0^4B_0^3)A_1^2B_2^2 \\
&\quad + (20304A_0^7B_0 - 9072A_0^4B_0^3)A_1^4A_2 \\
&\quad + (12780A_0^8 + 5400A_0^5B_0^2 + 540A_0^2B_0^4)A_1^4B_2 \\
&\quad \left. + (-11076A_0^6B_0 + 1512A_0^3B_0^3 - 36B_0^5)A_1^6 \right). \tag{A.14}
\end{aligned}$$

Note that  $a_n, b_n$  satisfy most of the properties of the  $W(E_8)$ -invariant Jacobi forms except the condition  $\mathbf{v}^2 \leq 2ml$  in the Fourier expansion.  $a_n, b_n$  are of index  $n$  and weight  $4-6n, 6-6n$ , respectively. It is useful to let the variables  $u, x, y$  transform formally as Jacobi forms of weights  $-6, -10, -15$  and index  $1, 2, 3$ , respectively. The whole curve then transforms as a Jacobi form of weight  $-30$  and index  $6$ .  $f, g$  are of weight  $-20, -30$  and index  $4, 6$ , respectively.

## B. Derivative formulas

$$q \frac{d}{dq} \ln \Delta = E_2, \quad (\text{B.1})$$

$$q \frac{d}{dq} E_2 = \frac{1}{12} (E_2^2 - E_4), \quad (\text{B.2})$$

$$q \frac{d}{dq} E_4 = \frac{1}{3} (E_4 E_2 - E_6), \quad (\text{B.3})$$

$$q \frac{d}{dq} E_6 = \frac{1}{2} (E_6 E_2 - E_4^2). \quad (\text{B.4})$$

$$(\partial_\xi t)_u = -2t^2, \quad (\text{B.5})$$

$$(\partial_\xi \omega)_u = 2\omega t, \quad (\text{B.6})$$

$$(\partial_\xi \phi)_u = 2\partial_\phi^{-1} t, \quad (\text{B.7})$$

$$\left( \partial_\xi \ln \tilde{\Delta} \right)_u = 24t, \quad (\text{B.8})$$

$$\partial_\xi \tilde{E}_{2k} = 4kt \tilde{E}_{2k} + 24\delta_{1,k}. \quad (\text{B.9})$$

$$(\partial_\xi (\partial_\phi^n \ln \omega))_u = -2 \sum_{k=0}^{n-1} \binom{n}{k+1} \partial_\phi^k t \partial_\phi^{n-k} \ln \omega + 2\partial_\phi^n t, \quad (\text{B.10})$$

$$(\partial_\xi (\partial_\phi^n t))_u = -2 \sum_{k=0}^{n-1} \left[ \binom{n}{k+1} + 2 \binom{n-1}{k} \right] \partial_\phi^k t \partial_\phi^{n-k} t \quad (n \geq 1). \quad (\text{B.11})$$

### C. Genus three amplitude

$$\begin{aligned}
F_3 = & (\partial_\phi^4 \ln \omega) \left( \frac{1}{2304} \tilde{E}_2^2 + \frac{1}{2592} \tilde{E}_4 \right) \\
& + (\partial_\phi^3 \ln \omega) (\partial_\phi \ln \omega) \left( \frac{1}{1152} \tilde{E}_2^2 - \frac{1}{6912} \tilde{E}_4 \right) \\
& + (\partial_\phi^2 \ln \omega)^2 \left( \frac{1}{2304} \tilde{E}_2^2 + \frac{11}{20736} \tilde{E}_4 \right) \\
& + (\partial_\phi^2 \ln \omega) (\partial_\phi \ln \omega)^2 \left( \frac{1}{2304} \tilde{E}_2^2 + \frac{1}{20736} \tilde{E}_4 \right) \\
& + (\partial_\phi^3 \ln \omega) (\partial_\phi t) \left( \frac{1}{41472} \tilde{E}_2^3 + \frac{11}{82944} \tilde{E}_4 \tilde{E}_2 - \frac{13}{82944} \tilde{E}_6 \right) \\
& + (\partial_\phi^2 \ln \omega) (\partial_\phi \ln \omega) (\partial_\phi t) \left( \frac{1}{13824} \tilde{E}_2^3 - \frac{25}{248832} \tilde{E}_4 \tilde{E}_2 + \frac{7}{248832} \tilde{E}_6 \right) \\
& + (\partial_\phi \ln \omega)^3 (\partial_\phi t) \left( \frac{1}{41472} \tilde{E}_2^3 - \frac{1}{31104} \tilde{E}_4 \tilde{E}_2 + \frac{1}{124416} \tilde{E}_6 \right) \\
& + (\partial_\phi^2 \ln \omega) (\partial_\phi^2 t) \left( \frac{7}{62208} \tilde{E}_4 \tilde{E}_2 - \frac{7}{62208} \tilde{E}_6 \right) \\
& + (\partial_\phi^2 \ln \omega) (\partial_\phi t)^2 \left( -\frac{1}{331776} \tilde{E}_2^4 + \frac{79}{1492992} \tilde{E}_4 \tilde{E}_2^2 - \frac{35}{373248} \tilde{E}_6 \tilde{E}_2 + \frac{131}{2985984} \tilde{E}_4^2 \right) \\
& + (\partial_\phi \ln \omega)^2 (\partial_\phi^2 t) \left( \frac{5}{248832} \tilde{E}_4 \tilde{E}_2 - \frac{5}{248832} \tilde{E}_6 \right) \\
& + (\partial_\phi \ln \omega)^2 (\partial_\phi t)^2 \left( -\frac{1}{331776} \tilde{E}_2^4 + \frac{43}{2985984} \tilde{E}_4 \tilde{E}_2^2 - \frac{25}{1492992} \tilde{E}_6 \tilde{E}_2 + \frac{1}{186624} \tilde{E}_4^2 \right) \\
& + (\partial_\phi \ln \omega) (\partial_\phi^3 t) \left( -\frac{1}{13824} \tilde{E}_2^3 + \frac{1}{19440} \tilde{E}_4 \tilde{E}_2 + \frac{13}{622080} \tilde{E}_6 \right) \\
& + (\partial_\phi \ln \omega) (\partial_\phi^2 t) (\partial_\phi t) \left( -\frac{5}{165888} \tilde{E}_2^4 + \frac{35}{497664} \tilde{E}_4 \tilde{E}_2^2 - \frac{5}{248832} \tilde{E}_6 \tilde{E}_2 - \frac{5}{248832} \tilde{E}_4^2 \right) \\
& + (\partial_\phi \ln \omega) (\partial_\phi t)^3 \left( -\frac{1}{497664} \tilde{E}_2^5 + \frac{29}{2985984} \tilde{E}_4 \tilde{E}_2^3 \right. \\
& \quad \left. - \frac{1}{110592} \tilde{E}_6 \tilde{E}_2^2 - \frac{1}{995328} \tilde{E}_4^2 \tilde{E}_2 + \frac{7}{2985984} \tilde{E}_6 \tilde{E}_4 \right) \\
& + (\partial_\phi^4 t) \left( -\frac{1}{20736} \tilde{E}_2^3 - \frac{121}{1244160} \tilde{E}_4 \tilde{E}_2 - \frac{173}{8709120} \tilde{E}_6 \right) \\
& + (\partial_\phi^3 t) (\partial_\phi t) \left( -\frac{11}{497664} \tilde{E}_2^4 - \frac{287}{3732480} \tilde{E}_4 \tilde{E}_2^2 + \frac{421}{6531840} \tilde{E}_6 \tilde{E}_2 + \frac{361}{10450944} \tilde{E}_4^2 \right) \\
& + (\partial_\phi^2 t)^2 \left( -\frac{1}{55296} \tilde{E}_2^4 - \frac{19}{331776} \tilde{E}_4 \tilde{E}_2^2 + \frac{19}{387072} \tilde{E}_6 \tilde{E}_2 + \frac{61}{2322432} \tilde{E}_4^2 \right) \\
& + (\partial_\phi^2 t) (\partial_\phi t)^2 \left( -\frac{13}{1990656} \tilde{E}_2^5 - \frac{1}{27648} \tilde{E}_4 \tilde{E}_2^3 \right. \\
& \quad \left. + \frac{25}{331776} \tilde{E}_6 \tilde{E}_2^2 - \frac{19}{1990656} \tilde{E}_4^2 \tilde{E}_2 - \frac{23}{995328} \tilde{E}_6 \tilde{E}_4 \right) \\
& + (\partial_\phi t)^4 \left( -\frac{7}{23887872} \tilde{E}_2^6 - \frac{181}{71663616} \tilde{E}_4 \tilde{E}_2^4 + \frac{19}{2239488} \tilde{E}_6 \tilde{E}_2^3 \right. \\
& \quad \left. - \frac{47}{7962624} \tilde{E}_4^2 \tilde{E}_2^2 - \frac{1}{559872} \tilde{E}_6 \tilde{E}_4 \tilde{E}_2 + \frac{73}{71663616} \tilde{E}_4^3 + \frac{1}{995328} \tilde{E}_6^2 \right).
\end{aligned} \tag{C.1}$$



## D. Conventions

We define the Eisenstein series, the modular discriminant and the  $j$ -invariant by their Fourier expansion

$$E_{2n}(\tau) = 1 + \frac{(2\pi i)^{2n}}{(2n-1)!\zeta(2n)} \sum_{k=1}^{\infty} \frac{k^{2n-1}q^k}{1-q^k}, \quad q = e^{2\pi i\tau}, \quad (\text{D.1})$$

$$\Delta(\tau) = q \left[ \prod_{k=1}^{\infty} (1-q^k) \right]^{24} = \frac{1}{1728} (E_4(\tau)^3 - E_6(\tau)^2), \quad (\text{D.2})$$

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)}. \quad (\text{D.3})$$

We often omit the argument of these functions, as far as it is  $\tau$ . When the argument is  $\tilde{\tau}$ , we use the following abbreviations

$$\tilde{E}_{2n} := E_{2n}(\tilde{\tau}), \quad \tilde{\Delta} := \Delta(\tilde{\tau}), \quad \tilde{j} := j(\tilde{\tau}). \quad (\text{D.4})$$

The Weierstrass  $\wp$ -function is defined as

$$\wp(z|2\pi\omega, 2\pi\omega\tau) = \frac{1}{z^2} + \sum_{m,n \in \mathbb{Z}_{\neq(0,0)}^2} \left[ \frac{1}{(z - \Omega_{m,n})^2} - \frac{1}{\Omega_{m,n}^2} \right], \quad \Omega_{m,n} = 2\pi\omega(m + n\tau). \quad (\text{D.5})$$

This function satisfies the following differential equation

$$(\partial_z \wp)^2 = 4\wp^3 - \frac{E_4(\tau)}{12\omega^4} \wp - \frac{E_6(\tau)}{216\omega^6}. \quad (\text{D.6})$$

The Jacobi theta functions are defined as

$$\vartheta_1(z, \tau) = i \sum_{n \in \mathbb{Z}} (-1)^n y^{n-1/2} q^{(n-1/2)^2/2}, \quad (\text{D.7})$$

$$\vartheta_2(z, \tau) = \sum_{n \in \mathbb{Z}} y^{n-1/2} q^{(n-1/2)^2/2}, \quad (\text{D.8})$$

$$\vartheta_3(z, \tau) = \sum_{n \in \mathbb{Z}} y^n q^{n^2/2}, \quad (\text{D.9})$$

$$\vartheta_4(z, \tau) = \sum_{n \in \mathbb{Z}} (-1)^n y^n q^{n^2/2}, \quad (\text{D.10})$$

where  $y = e^{2\pi iz}$ ,  $q = e^{2\pi i\tau}$ . We also use the following abbreviated notation

$$\vartheta_k(\tau) := \vartheta_k(0, \tau). \quad (\text{D.11})$$

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